

Computation of Cutoff Wavenumbers of TE and TM Modes in Waveguides of Arbitrary Cross Sections Using a Surface Integral Formulation

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Abstract—This paper describes a procedure for obtaining the cutoff wavenumbers of TE and TM modes in waveguides of arbitrary cross section. A surface integral equation approach is used where the E -field equation has been transformed into a matrix equation using the method of moments. An iterative technique has been used to pick the eigenvalues of the solution matrix which corresponds to the waveguide cutoff wavenumbers. The salient features of this technique are its speed, its simplicity, and the absence of any spurious modes while treating waveguides of arbitrary cross section. The first four modes have been tabulated for various waveguides and results are in very good agreement with published data.

I. INTRODUCTION

ANALYSIS OF hollow conducting waveguides is an area that has been in the limelight lately. Various methods to find the cutoff wavenumbers are available in the literature, among them the finite difference and integral operator methods. A surface integral equation approach has been used here because of its simplicity and generality.

Treatment of waveguides with arbitrary cross section is a challenging task to solve on the computer. Though numerous approaches are available in the literature to solve this problem, they may be deficient in two areas, viz. speed and simplicity, for simulating waveguides of arbitrary cross section. As an example, the finite difference method may be a very difficult way of treating waveguides with nonrectangular cross sections. The use of the integral equation method to solve for the waveguide cutoff wavenumbers has been explained here by placing considerable importance on the generality of the problem. Spielman and Harrington [1] have treated this problem by using a similar approach. Their method, however, leads to the existence of spurious modes. The method used in this paper eradicates the existence of spurious modes and converges quadratically in the vicinity of a root (hence is faster). The cutoff wavenumbers of a coaxial waveguide with cylindrical outer conductor and rectangular inner conductor have been computed here, which serves as an exam-

ple to verify the speed and simplicity of this method. The CPU time for picking the dominant mode has been provided to give an idea of the speed of this method. The algorithm was run on an ALLIANT FX/80, a vector concurrent computer. However, only a single processor in scalar mode was used, whose speed is comparable to that of a VAX 8530.

II. INTEGRAL EQUATIONS

The hollow conducting waveguides are assumed to be infinite in the z direction and to have arbitrary cross section. The waveguide is completely filled with a homogeneous dielectric (air in this case). The matrix containing the information on the mode cutoff wavenumbers can be obtained by generating the matrix corresponding to the scattering problem for the same structure under plane wave illumination. Consider an equivalent problem where E^i is a wave incident on the hollow conducting waveguide. This generates surface currents J on the body that in turn reradiate into free space. Since the body is a conductor, the total tangential electric field vanishes on the surface of the body. In other words,

$$\mathbf{n} \times [\mathbf{E}^i + \mathbf{E}^s] = 0 \quad \text{on } C, \quad (1)$$

where E^s is the scattered field produced by the current J , and \mathbf{n} is the unit vector normal to the surface of the body.

At cutoff, the surface currents J on the walls of the waveguide produce zero fields on the contour C . Hence

$$\mathbf{n} \times \mathbf{E}^s = 0. \quad (2)$$

This is the homogeneous E -field integral equation which can be used in evaluating the cutoff wavenumbers of a waveguide. The equation for E^s depends on whether the fields in the waveguide are TM or TE to the z axis.

A. TM Case

The scattered field E^s and the surface electric currents are in the z direction. Since the current J is independent of z there is no electric charge associated with the surface current. Equation (2) then simplifies to

$$\mathbf{z} \cdot [-j\omega A(\mathbf{J})] = 0 \quad (3)$$

where \mathbf{z} is the unit vector in the z direction, ω is the

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III. EVALUATION OF CUTOFF WAVENUMBERS

The wavenumber k appears as an argument in (6) and (10). The problem therefore reduces to finding the wavenumbers of the $[Z]$ matrix that represent the cutoff modes. At cutoff, the matrix Z becomes singular. Hence

$$\det[Z] = 0. \quad (12)$$

The determinant of a matrix Z requires $\theta(n^3)$ operations. However (12) can be rewritten as

$$\lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n = 0 \quad (13)$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the matrix Z . This is a very efficient way of evaluating the determinant since only $\theta(n^2)$ operations are necessary. Since the minimum eigenvalue listed in (13) approaches zero at cutoff, (11) becomes

$$[Z][J] = \lambda_{\min}[J] \quad (14)$$

where λ_{\min} is the smallest eigenvalue of Z . Hence the problem reduces to finding the wavenumber at which the minimum eigenvalue of $[Z]$ goes to zero. The matrix $[Z]$ is unsymmetric and complex and the eigenvalues are also complex. The absolute value of the minimum eigenvalue has been used in the algorithm for finding the cutoff wavenumbers. A very expensive way of searching for the cutoff wavenumbers is by using a scanning procedure wherein the minimum eigenvalues are computed for a certain range of frequencies. The wavenumber at which the minimum eigenvalue is the smallest is then the cutoff wavenumber of the waveguide. This method has two drawbacks, namely, the evaluation of the Z matrix over a range of frequencies and the inaccuracy in the result obtained which depends on the scanning step. These drawbacks have been solved by using a method expounded by Muller [2]. Muller's method is an iterative technique which converges quadratically in the vicinity of a root, does not require the evaluation of any derivatives, and obtains complex roots even when these roots are not simple.

IV. ALGORITHM FOR OBTAINING CUTOFF WAVENUMBERS

The steps involved are given below,

- i) Choose a lower limit for the wavenumber and use this information to patch the waveguide using the necessary number of expansion functions.
- ii) An initial guess for the wavenumber is chosen and the Z matrix is evaluated at this frequency. The integral equation corresponding to either the TM or the TE case is used for generating the Z matrix.
- iii) The eigenvalues of the Z matrix are then found using an IMSL routine. These eigenvalues are varied to find the minimum absolute eigenvalue, which is then used as the function to be minimized.
- iv) Muller's method is used iteratively to choose the wavenumbers that minimize the function. Steps (ii) and (iii) are repeated at each wavenumber chosen iteratively by Muller's method. The wavenumber corresponding to the minimum value of the function is the cutoff wavenumber of the waveguide. An error criterion has been used to terminate the algorithm,

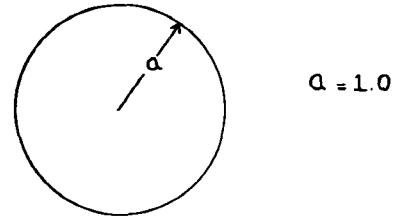


Fig. 2. Cylindrical waveguide.

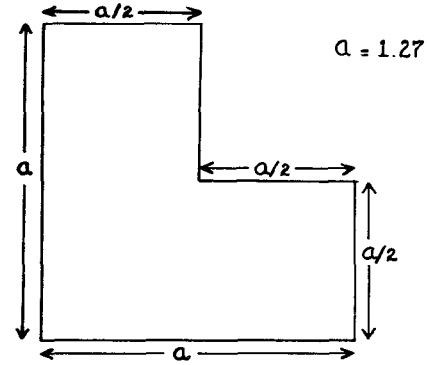


Fig. 3. L-shaped rectangular waveguide.

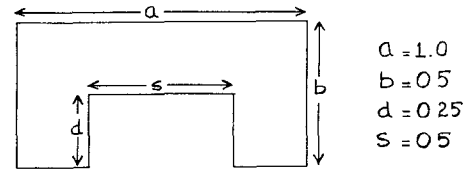


Fig. 4. Single ridge waveguide.

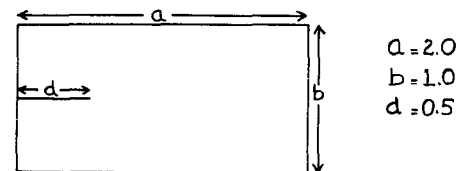


Fig. 5. Vane rectangular waveguide.

which is

$$|k_{n+1} - k_n|/k_n < 10^{-5}$$

where k_{n+1} is the present estimate and k_n is the past estimate for the wavenumbers.

V. EXAMPLES

The first four cutoff wavenumbers have been evaluated for various waveguides as shown in Figs. 2–8. Tables I(a)–VII(a) and I(b)–VII(b) provide the first four cutoff wavenumbers for the waveguides depicted in Figs. 2–8 for the TM and TE modes, respectively.

VI. POINTS TO NOTE

- 1) On an average, approximately five expansion functions per wavelength are necessary to simulate the structure being analyzed.
- 2) Though Muller's method is a fast iterative technique, the rate of convergence depends on the initial guess, which in turn affects the CPU time. The dominant mode of each waveguide has therefore been computed using an initial

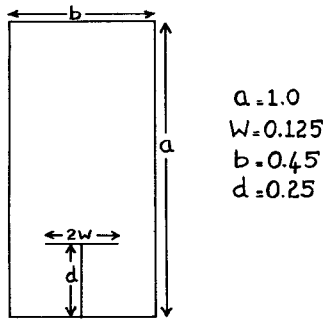


Fig. 6. T-septate rectangular waveguide.

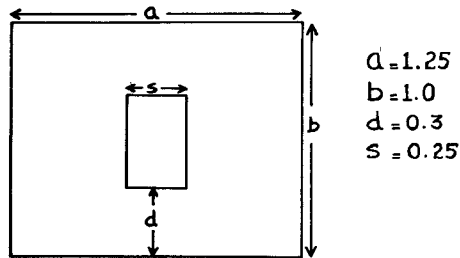


Fig. 7. Coaxial rectangular waveguide.

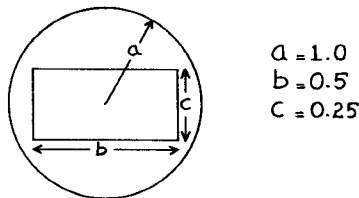


Fig. 8. Coaxial waveguide.

TABLE I(a)
CYLINDRICAL WAVEGUIDE (TM CASE): THE FIRST FOUR CUTOFF WAVENUMBERS FOR A WAVEGUIDE WITH CYLINDRICAL CROSS SECTION GIVE AN IDEA OF THE ACCURACY OF THIS METHOD.

Num	$k_c(\text{exact})$	$k_c(\text{computed})$	Difference%
1	2.4048	2.4111	0.26
2	3.8317	3.8416	0.26
3	5.1356	5.1485	0.25
4	5.5200	5.5346	0.26

Order of Matrix = 40

Cpu Time for Dominant Mode = 20 secs

TABLE I(b)
CYLINDRICAL WAVEGUIDE (TE CASE): A CHECK TO DETERMINE THE ACCURACY OF THIS METHOD WITH THE FIRST FOUR MODES LISTED.

Num	$k_c(\text{exact})$	$k_c(\text{computed})$	Difference%
1	1.8412	1.8462	0.27
2	3.0542	3.0645	0.34
3	3.8317	3.8422	0.27
4	4.2012	4.2200	0.45

Order of Matrix = 40

Cpu Time for Dominant Mode = 136 secs

TABLE II(a)
L-SHAPED WAVEGUIDE (TM CASE)

Num	k_c	Matrix Size
1	4.8677	40
2	6.1361	40
3	6.9908	40
4	8.5525	60

 k_c for dominant mode = 4.819 [Reid and Walsh]

Cpu Time for Dominant Mode = 45 secs

Difference in dominant mode = 1%

TABLE II(b)
L-SHAPED WAVEGUIDE (TE CASE)

Num	k_c	Matrix Size
1	1.8917	14
2	2.9159	14
3	4.8755	22
4	5.2463	22

 k_c for dominant mode = 1.88 [Sarkar et al]

Cpu Time for Dominant Mode = 29 secs

Difference in dominant mode = 0.62%

TABLE III(a)
SINGLE RIDGE WAVEGUIDE (TM CASE)

Num	k_c	Matrix Size
1	12.0381	40
2	12.2938	40
3	13.9964	70
4	15.5871	70

 k_c for dominant mode = 12.164 [Spielman and Harrington]

Cpu Time for Dominant Mode = 30 secs

Difference in dominant mode = 1%

TABLE III(b)
SINGLE RIDGE WAVEGUIDE (TE CASE)

Num	k_c	Matrix Size
1	2.2496	28
2	4.9436	28
3	6.5189	28
4	7.5642	28

 k_c for dominant mode = 2.2566 [Sarkar et al]

Cpu Time for Dominant Mode = 156 secs

Difference in dominant mode = 0.31%

guess of $k_c = 1.3$ to give some estimate of the CPU time involved.

3) The surface integral equation method has been used by Spielman and Harrington in [1]. An assumption that has been used in it is that at resonance the problem reduces to finding the wavenumbers that make the imaginary part of the $[Z]$ matrix singular. This, however, produces spurious modes. Hence it is worthwhile to note that the use of $[Z]$ instead of $\text{Im}[Z]$ eliminates these spurious modes.

TABLE IV(a)
VANED RECTANGULAR WAVEGUIDE (TM CASE)

Num	k_c	Matrix Size
1	3.6770	50
2	4.9279	50
3	6.4151	50
4	7.0220	50

k_c for dominant mode = 3.65 [Sarkar et al]
 Cpu Time for Dominant Mode = 23 secs
 Difference in dominant mode = 0.74%

TABLE IV(b)
VANED RECTANGULAR WAVEGUIDE (TE CASE): THE WAVEGUIDE HAS BEEN MODELED IN SUCH A WAY THAT KIRCHHOFF'S CURRENT LAW IS SATISFIED AT THE JUNCTION.

Num	k_c	Matrix Size
1	1.5695	40
2	2.1156	40
3	3.1568	40
4	3.3046	40

k_c for dominant mode = 1.57 [Sarkar et al]
 Cpu Time for Dominant Mode = 328 secs
 Difference in dominant mode = 0.03%

TABLE V(a)
T-SEPTATE WAVEGUIDE (TM CASE)

Num	k_c	Matrix Size
1	8.1293	36
2	10.8659	36
3	14.3161	46
4	14.5550	46

k_c for dominant mode = 8.12 [Sarkar et al]
 Cpu Time for Dominant Mode = 19.74 secs
 Difference in dominant mode = 0.11%

TABLE V(b)
T-SEPTATE WAVEGUIDE (TE CASE): THE WAVEGUIDE HAS BEEN MODELED IN SUCH A WAY THAT KIRCHHOFF'S CURRENT LAW IS SATISFIED AT THE T JUNCTION.

Num	k_c	Matrix Size
1	2.9752	40
2	3.1677	40
3	5.6535	66
4	7.2357	66

k_c for dominant mode = 3.0 [Sarkar et al]
 Cpu Time for Dominant Mode = 169 secs
 Difference in dominant mode = 0.83%

TABLE VI(a)
COAXIAL RECTANGULAR WAVEGUIDE (TM CASE)

Num	k_c	Matrix Size
1	6.8400	34
2	8.3286	48
3	8.4905	48
4	10.2959	48

k_c for dominant mode = 6.8650 [Gruner]
 Cpu Time for Dominant Mode = 40 secs
 Difference in dominant mode = 0.36%

TABLE VI(b)
COAXIAL RECTANGULAR WAVEGUIDE (TE CASE)

Num	k_c	Matrix Size
1	2.0789	24
2	2.8465	24
3	3.9631	24
4	5.2864	26

k_c for dominant mode = 2.0774 [Gruner]
 Cpu Time for Dominant Mode = 57 secs
 Difference in dominant mode = 0.07%

TABLE VII(a)
COAXIAL WAVEGUIDE (TM CASE)

Num	k_c	Matrix Size
1	3.8919	56
2	4.1666	56
3	4.4450	56
4	5.2645	56

k_c for dominant mode = published data not available
 Cpu Time for Dominant Mode = 51 secs

TABLE VII(b)
COAXIAL WAVEGUIDE (TE CASE)

Num	k_c	Matrix Size
1	1.7407	56
2	3.0441	56
3	4.2199	56
4	4.6451	56

k_c for dominant mode = published data not available
 Cpu Time for Dominant Mode = 447 secs

4) The code developed treats waveguides with arbitrary cross section. The input data necessary to model the waveguide are all that is required from the user. This makes the algorithm simple to use while maintaining its generality.

5) Due to the absence of charge on a TM waveguide, the computation of the Z matrix is much faster for the TM case than for the TE case. Hence the computation of the TE cutoff modes takes more time, as can be seen from the tabulation.

6) For the TE case (for example, vane and T-septate rectangular waveguides), it is necessary to densely place expansion functions near a junction so as to better simulate Kirchhoff's current law.

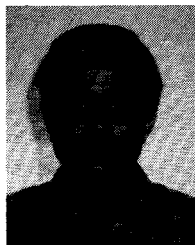
7) Muller's method can be used to find a prescribed number of zeros, real or complex, of an arbitrary function. Hence, this method can be used to pick multiple cutoff wavenumbers of a waveguide.

VII. SUMMARY AND CONCLUSIONS

A fast iterative technique has been developed to compute the cutoff wavenumbers of conducting waveguides with arbitrary cross section. The integral equation approach used here eradicates the existence of any spurious modes. Both TE and TM cases have been considered. The method is being extended to waveguides partially filled with dielectrics, by using the surface equivalence principle.

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